

## Page 17 Derivation of (2-28), the Gell-Mann $SU(3)$ matrix. by Luc Longtin

We recall that this matrix (2-28) is valid only in the limit of *small independent parameters*.

We start from the totally general 3X3 complex matrix in the form of (2-26), to which we impose the conditions (2-27) for a *special unitary group*. With a view of writing the matrix in a form that will reduce to the identity matrix in the limit where the independent parameters vanish, we write:

$$N = \begin{bmatrix} 1 + \delta_1 & \delta_4 & \delta_5 \\ \delta_7 & 1 + \delta_2 & \delta_6 \\ \delta_8 & \delta_9 & 1 + \delta_3 \end{bmatrix} \quad \dots \text{with conditions } N^\dagger N = I \text{ and } \text{Det } N = 1$$

From the *special* condition,  $\text{Det } N = 1$ , we must have:

$$(1 + \delta_1)[(1 + \delta_2)(1 + \delta_3) - \delta_6\delta_9] - \delta_4[\delta_7(1 + \delta_3) - \delta_6\delta_8] + \delta_5[\delta_7\delta_9 - \delta_8(1 + \delta_2)] = 1$$

Now, this equation must be valid for *all* values of the variable parameters. In particular, it must *also* be valid in the limit of *small* parameters. We can use this fact to find relations that must be satisfied among the parameters *in that limit*. Thus, keeping *only first-order* terms, we get:

$$(1 + \delta_1)(1 + \delta_2)(1 + \delta_3) \approx (1 + \delta_1)(1 + \delta_2 + \delta_3) \approx 1 + \delta_1 + \delta_2 + \delta_3 = 1$$

$$\text{So: } \delta_1 + \delta_2 + \delta_3 \approx 0$$

From the *unitary* condition,  $N^\dagger N = I$ , we must thus have:

$$N^\dagger N = \begin{bmatrix} 1 + \delta_1^* & \delta_7^* & \delta_8^* \\ \delta_4^* & 1 + \delta_2^* & \delta_9^* \\ \delta_5^* & \delta_6^* & 1 + \delta_3^* \end{bmatrix} \begin{bmatrix} 1 + \delta_1 & \delta_4 & \delta_5 \\ \delta_7 & 1 + \delta_2 & \delta_6 \\ \delta_8 & \delta_9 & 1 + \delta_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Again, this equation must be valid for *all* values of the variable parameters. In particular, it must *also* be valid in the limit of *small* parameters. Thus, keeping *only first-order* terms, we get:

Element      Expression

1-1	$(1 + \delta_1^*)(1 + \delta_1) \approx 1 + \delta_1 + \delta_1^* \approx 1$	so	$\delta_1 + \delta_1^* \approx 0$		
2-2	$(1 + \delta_2^*)(1 + \delta_2) \approx 1 + \delta_2 + \delta_2^* \approx 1$	so	$\delta_2 + \delta_2^* \approx 0$		
3-3	$(1 + \delta_3^*)(1 + \delta_3) \approx 1 + \delta_3 + \delta_3^* \approx 1$	so	$\delta_3 + \delta_3^* \approx 0$		
1-2	$\delta_4 + \delta_7^* \approx 0$	1-3	$\delta_5 + \delta_8^* \approx 0$	2-3	$\delta_6 + \delta_9^* \approx 0$
2-1	$\delta_4^* + \delta_7 \approx 0$	3-1	$\delta_5^* + \delta_8 \approx 0$	3-2	$\delta_6^* + \delta_9 \approx 0$

The first three results tell us that  $\delta_1, \delta_2, \delta_3$  are *pure imaginary* complex numbers. Making use of the last six results (three of which are independent), and the special condition, we can write:

$$N = \begin{bmatrix} 1 + \delta_1 & \delta_4 & \delta_5 \\ -\delta_4^* & 1 + \delta_2 & \delta_6 \\ -\delta_5^* & -\delta_6^* & 1 - (\delta_1 + \delta_2) \end{bmatrix} \quad \dots \text{a friendlier form than (2-28)!}$$

NOTE: This matrix has *eight* independent (real) parameters, namely the real and imaginary parts of  $\delta_4, \delta_5, \delta_6$  (which count for *six*) and the pure imaginary  $\delta_1, \delta_2$  (which count for *two*).

We can reproduce Gell-Mann's form, (2-28), via the following change of variables:

$$\delta_1 = i\alpha_3 + i\frac{\alpha_8}{\sqrt{3}} \quad \delta_2 = -i\alpha_3 + i\frac{\alpha_8}{\sqrt{3}} \quad \text{so} \quad \delta_1 + \delta_2 = i\frac{2\alpha_8}{\sqrt{3}}$$

NOTE:  $\alpha_3, \alpha_8$  are both *real* parameters; this ensures that  $\delta_1, \delta_2$  are both pure imaginary

$$\delta_4 = \alpha_2 + i\alpha_1 \quad \delta_5 = \alpha_5 + i\alpha_4 \quad \delta_6 = \alpha_7 + i\alpha_6$$

NOTE: In these relations, *all* the  $\alpha$  variables are *real* parameters. Therefore, we can write:

$$-\delta_4^* = -\alpha_2 + i\alpha_1 \quad -\delta_5^* = -\alpha_5 + i\alpha_4 \quad -\delta_6^* = -\alpha_7 + i\alpha_6$$

$$\text{So: } N = \begin{bmatrix} 1 + i\alpha_3 + i\frac{\alpha_8}{\sqrt{3}} & \alpha_2 + i\alpha_1 & \alpha_5 + i\alpha_4 \\ -\alpha_2 + i\alpha_1 & 1 - i\alpha_3 + i\frac{\alpha_8}{\sqrt{3}} & \alpha_7 + i\alpha_6 \\ -\alpha_5 + i\alpha_4 & -\alpha_7 + i\alpha_6 & 1 - i\frac{2\alpha_8}{\sqrt{3}} \end{bmatrix} \quad \dots \text{factoring out a factor of } i \dots$$

$$\text{Or: } N = i \begin{bmatrix} -i + \alpha_3 + \frac{\alpha_8}{\sqrt{3}} & \alpha_1 - i\alpha_2 & \alpha_4 - i\alpha_5 \\ \alpha_1 + i\alpha_2 & -i - \alpha_3 + \frac{\alpha_8}{\sqrt{3}} & \alpha_6 - i\alpha_7 \\ \alpha_4 + i\alpha_5 & \alpha_6 + i\alpha_7 & -i - \frac{2\alpha_8}{\sqrt{3}} \end{bmatrix}$$

$$\text{Or: } N = N(0) + i \begin{bmatrix} \alpha_3 + \frac{\alpha_8}{\sqrt{3}} & \alpha_1 - i\alpha_2 & \alpha_4 - i\alpha_5 \\ \alpha_1 + i\alpha_2 & -\alpha_3 + \frac{\alpha_8}{\sqrt{3}} & \alpha_6 - i\alpha_7 \\ \alpha_4 + i\alpha_5 & \alpha_6 + i\alpha_7 & -\frac{2\alpha_8}{\sqrt{3}} \end{bmatrix} \quad \dots \text{where } N(0) = I, \text{ the identity}$$

The advantage of this form is that *all* the  $\alpha$  parameters are *real*.

NOTE: We recall that this derivation of the matrix representation for the  $SU(3)$  group is valid only in the limit of *small* independent parameters. That is, for  $|\alpha_i| \ll 1$ .